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Oscillations over basins of variable depth

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This paper is concerned with oscillations of a body of water in basins of variable depth, employing a system of linearized equations which can be obtained from the theory of a directed fluid sheet for an incompressible, homogeneous, inviscid fluid (Green & Naghdi 1976*a*, 1977). For free oscillations over a level bottom, an assessment of the range of validity of the linearized theory is made by an appropriate comparison with a corresponding well-known exact solution (Lamb 1932). This assessment indicates an 'intermediate' range of validity for the linearized theory not covered by usual classical approximations for long waves. Encouraged by this assessment, we apply the linear theory of a directed fluid sheet to basins of variable depth; and, in particular, consider a class of basin profiles whose equilibrium depth (along its width) varies in one direction only. By a method of asymptotic integration, a general solution is obtained which is relatively simple and accounts for the effect of vertical inertia. The solution is sinusoidal in time, periodic along the breadth direction and involves Bessel functions of the first order in the width direction. For two special basin profiles, detailed comparisons are made between the predictions of the asymptotic solution (i.e. the frequencies in the lowest modes of oscillations) with corresponding results obtained by other procedures.

1. Introduction

This paper is concerned with small-amplitude, free-surface oscillations over basins of variable depth. Particular attention is given to oscillations over a class of basins whose plan view is rectangular and whose equilibrium depth varies in one direction only. A description of such a basin is given in the first paragraph of §2 and a sketch of a basin of this kind in the (x,z)-plane of the rectangular Cartesian coordinates (x,y,z) with its equilibrium height varying with x is shown in figure 1.

In the development that follows, we utilize the linearized version of the differential equations of the restricted theory of a directed fluid sheet for an incompressible, homogeneous, inviscid fluid in the form derived by Green & Naghdi (1976*a*, 1977). The development of the basic equations of this theory are based on a simple physical model whose main kinematical ingredients may be viewed as corresponding to an assumption for the velocity $\boldsymbol{v}^*$ in the three-dimensional theory such that the horizontal components of $\boldsymbol{v}^*$ are independent of z and its vertical component is linear in z (Green & Naghdi 1976*b*). Moreover, the mode of derivation in the direct approach (Green & Naghdi 1976*a*, 1977) is substantially different from standard procedures in which approximate theories are obtained from the three-dimensional equations of fluid dynamics by different procedures. Because of this, it is worth emphasizing that, in the context of the topic discussed here, important features of the direct approach (for construction of an approximate theory) are (i) the satisfaction of the conservation of mass and linear momentum (in the three-dimensional theory), if not pointwise, at

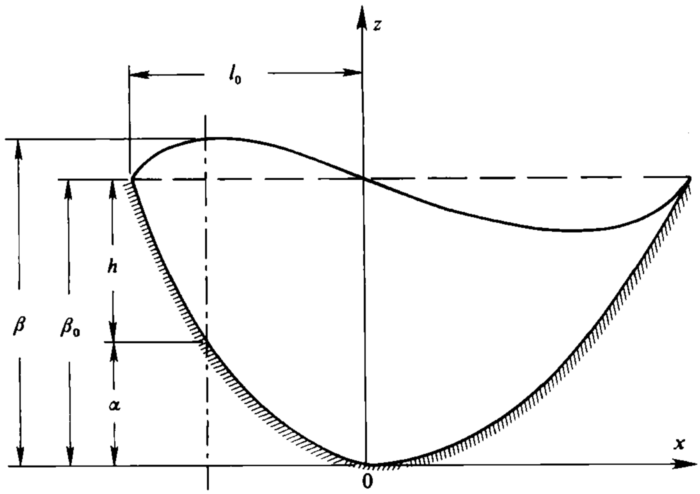

FIGURE 1. A sketch of a basin containing a fluid of variable depth showing its vertical cross-section perpendicular to y-direction of the fixed rectangular Cartesian coordinates (x, y, z) with origin at the lowest elevation point of the basin. Also indicated is the vertical location β_0 of the undisturbed level of the top free surface of the fluid (shown by a horizontal dashed line), the vertical location α of the bottom surface of the fluid, the equilibrium depth $h = \beta_0 - \alpha$ of the fluid at the location $(-x)$ and a characteristic length l_0 in the undisturbed free surface.

least in the sense of suitable weighted averages of these conservation laws, and (ii) the satisfaction of the surface (or boundary) conditions at the top and bottom surfaces of the fluid sheet in terms of the assumed kinematics.

Intuitively, one might expect that an approximate linear theory such as that of a directed fluid sheet (Green & Naghdi 1976*a*, 1977) – which restricts the director to remain parallel to a fixed direction for all time – would be applicable only to situations in which the value of a parameter $\mu = l_0/\beta_0$ (see figure 1) is large and all horizontal gradients are small compared to vertical gradients. However, as will be discussed presently, certain aspects of the theory pertaining to the prediction of the wave height and the frequency of oscillation (or the dispersion relation) are much better than one would at first suppose and μ need not necessarily be large. In order to assess the nature of this contention, from the linear theory of a directed fluid sheet we obtain a dispersion relation, and by comparison of this with a corresponding exact dispersion relation given by Lamb (1932, p. 440) demonstrate that the theory of a directed fluid sheet has an 'intermediate' range of validity beyond the classical approximation for long waves.

Preparatory to this task and by way of background, we note that the linearized equations of a directed fluid sheet for an incompressible, homogeneous, inviscid fluid (Green & Naghdi 1976*a*, 1977) are reduced in §2 to a system of partial differential equations for the determination of the horizontal velocities u, v [see (2.13*a*, *b*)]. When the motion is confined to the (x, z)-plane (with $v = 0$), this system reduces to a differential equation in u (see (2.14)) which in the case of a level bottom reads as

$$u_{tt} - ghu_{xx} - \tfrac{1}{3}h^2 u_{ttxx} = 0. \tag{1.1}$$

In (1.1), the equilibrium depth h = constant, u is the horizontal velocity of the fluid in x-direction, g is the constant gravitational acceleration, t denotes time and subscripts indicate partial differentiation. In the absence of the effect of vertical inertia and with the pressure regarded only as hydrostatic, (1.1) reduces to that of the classical shallow-water theory for long-wave approximation (Lamb 1932, arts 189 and 193).

1.1. *An assessment of the range of validity of the linearized theory*

For time-harmonic oscillations with constant frequency ω, a comparison of dispersion relations obtained from (1.1) and the corresponding linearized KdV (Korteweg & de Vries 1895) and BBM (Benjamin, Bona & Mahony 1972) equations was given previously (Green, Laws & Naghdi 1974). Here we take up the matter further and note that the dispersion relation calculated from (1.1) is

$$c^2 = \left(\frac{\omega}{k}\right)^2 = \frac{gh}{1+\frac{1}{3}k^2h^2}, \tag{1.2}$$

where c is the phase velocity and k is the wavenumber. From (1.2), which holds over a level bottom, by taking the limit as $k \rightarrow \infty$, it follows that there exists a cutoff frequency ω_c given by

$$\omega_c^2 = \frac{3g}{h}. \tag{1.3}$$

Alternatively, this cutoff frequency can be obtained directly from (1.1) as a condition for the existence of oscillatory solution. In fact, the existence of a cutoff frequency is a characteristic feature of all linear approximate theories of the type under discussion which include (at least partially) the effect of vertical inertia. This is in contrast to the classical shallow-water equation, which does not include vertical inertia and has no cutoff frequency.

We now recall from Lamb (1932, art. 257, p. 440) the exact solution for the free oscillation of a limited mass of water in a basin of uniform depth. The frequency† ω^* (corresponding to σ in equation (4) of Lamb 1932, p. 440) is given by

$$\omega^{*2} = \frac{g}{h} kh \tanh kh, \tag{1.4}$$

where the wavenumber k is related to the wavelength λ^* by

$$k = \frac{2\pi}{\lambda^*}. \tag{1.5}$$

We now ask for what value of the wavelength in the exact theory (say for $\lambda^* = \lambda_1^*$) associated with (1.4) is the frequency ω^* equal to the cutoff frequency ω_c given by (1.3)? In order to answer this, we set

$$\omega^{*2} = \omega_c^2 \Rightarrow \frac{g}{h} kh \tanh kh = 3\frac{g}{h}, \tag{1.6}$$

from which (since $\tanh 3 \simeq 1$) we obtain $kh \simeq 3$. Hence, from (1.5) we have

$$\frac{\lambda_1^*}{h} = \frac{2\pi}{3} \simeq 2.09, \tag{1.7}$$

and we may conclude that:

(i) For $\lambda^* \leqslant \lambda_1^*$, $\omega^* \geqslant \omega_c$ and the prediction of the theory of a directed fluid sheet is *out of the range* of the exact theory.

† The use of the notations ω^* (and also λ^*) for frequency (and for wavelength) for the exact solution is to avoid confusion with the corresponding notations (ω, λ) in the present paper.

(ii) For $\lambda^* > \lambda_1^*$, $\omega^* < \omega_c$ and the prediction of the theory of a directed fluid sheet is *within the range* of the exact theory.

Clearly the lower bound for the wavelength λ in the theory of a directed fluid sheet, obtained from (1.6) and given by (1.7), is significantly different from the assumption $(\lambda^*/h) \gg 1$, usually associated with the classical theory (Lamb 1932) for long waves.

In addition to the observation concerning the range of validity of the approximate theory used in the present paper, it is also informative to provide a further discussion of the approximate dispersion relation (1.2) and the corresponding dispersion relation from the exact theory (Lamb 1932, p. 440). Making use of power-series expansion, the dispersion relation from the exact result (1.4) can be written as

$$\frac{c}{(gh)^{\frac{1}{2}}} = \{1-\tfrac{1}{3}(kh)^2+\tfrac{2}{15}(kh)^4-\ldots\}^{\frac{1}{2}}, \tag{1.8}$$

while the corresponding expression from (1.2) is

$$\frac{c}{(gh)^{\frac{1}{2}}} = \{1-\tfrac{1}{3}(kh)^2+\tfrac{1}{9}(kh)^4-\ldots\}^{\frac{1}{2}}. \tag{1.9}$$

The first term on the right-hand side of both (1.8) and (1.9) represents the value appropriate for long waves and is the same also for the leading term in the power-series expansion of the dispersion relations of other approximate equations such as those of the KdV and the BBM. An examination of the second terms on the right-hand sides of (1.8) and (1.9) immediately reveals that the dispersion relation (1.9) of the theory of a directed fluid sheet is a truncation of the second-order term of the exact dispersion relation and as such is much closer to the exact theory than either the KdV or the BBM dispersion relations. Even for a fairly short wavelength $(\lambda^*/h) = 2.5$, the percentage difference for the prediction of $c/(gh)^{\frac{1}{2}}$ from the dispersion relation (1.2) and the exact dispersion relation (1.4) is less than 10%.

1.2. *The scope of the paper*

The main purpose of the present paper is to obtain a fairly general solution of the problem of free-surface oscillations over basins of variable depth, which would include the effect of vertical inertia and which would remain valid also at an end point of the basin, while the basin's profile is left unspecified. The discussion in §1.1 clearly indicates an 'intermediate' range of validity for the application of the linear theory to fluid motions over a level bottom, and one would expect that a similar 'intermediate' range of validity holds also for a variable bottom. In this connection, it should be noted that a cutoff frequency of the same form as (1.3) exists also for variable bottoms, but with the constant h replaced with $h_{\max}$, i.e.

$$\omega_c^2 = 3\frac{g}{h_{\max}}. \tag{1.10}$$

Again, this is a characteristic feature of the differential equations ((2.13) and (2.14)) for variable h to be introduced subsequently; and this feature persists throughout the solutions obtained in §§3 and 4 and is further discussed in §5.

A quantitative assessment of the nature of linear theory with variable bottom is undertaken for two special basin profiles in §3, with the use of the differential equations of §2 for variable bottoms. In particular, for two-dimensional fluid motions in the (vertical) (x, z)-plane, we obtain series solutions for two symmetric basin profiles, one with a triangular bottom shown in figure 2 and another with a parabolic

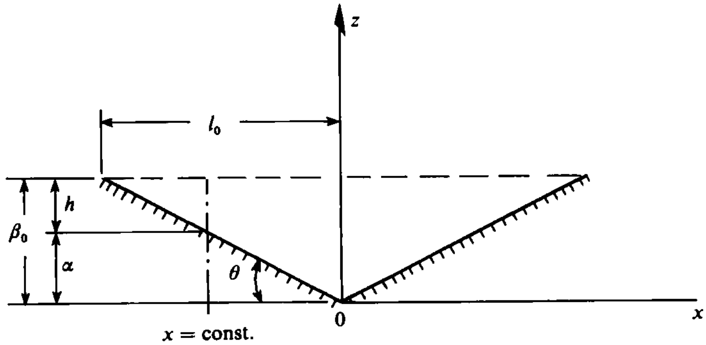

FIGURE 2. A sketch of a basin of constant slope whose cross-section (in the (x, z)-plane) is symmetrical about the z-axis and consists of two straight lines. Also indicated is the angle θ which each line makes with the horizontal, the half length l_0 in the undisturbed free surface, as well as several other quantities identified in the caption of figure 1.

bottom. In each case, we calculate the frequencies for the lowest antisymmetric and symmetric modes which are obtained from the first three orders of approximation (in the square of the frequency) of the series solutions. As might be expected, the results of the first approximations for both bottom profiles are the same as those of the classical results in which the effect of vertical inertia is neglected. Keeping this in mind, the results of the second and third approximations in the series solutions are then compared with those predicted by the first approximations, as well as the two known exact solutions (corresponding to the special values of angle θ in figure 2, i.e. $\theta = \frac{1}{4}\pi, \frac{1}{6}\pi$) for a triangular bottom due to Kirchhoff (1879) and Greenhill (1887) and a recent solution for a parabolic bottom obtained by Miles (1985) with the use of a variational approximation procedure.

In the rest of the paper (§§4, 5), attention is confined to a basin whose plan view (the (x, y)-plane) is rectangular. Thus, for a fairly general class of basin profiles whose equilibrium depth varies with one coordinate (in the x-direction) along the basin width, free oscillations of a body of water in the basin are considered in §4. These oscillations are sinusoidal in time and periodic (in the y-direction) along the breadth. By a method of asymptotic integration due to Langer (1935), a general oscillatory solution is obtained for the class of basin profiles mentioned above. The resulting solution has a relatively simple form and involves Bessel functions of the first order in the width direction [see (4.3), (4.7) and (4.11)].

We conclude the present paper with some calculations and remarks pertaining to applications of the general solution of §4. In particular, in §5 we again consider the free-surface oscillations in the (x, z)-plane of two symmetric basins with triangular and parabolic bottom profiles. For each of these two profiles, by application of the asymptotic solution of §4, we calculate the lowest frequencies in both the antisymmetric and symmetric modes. The results of these calculations agree rather well with those obtained from the third approximations in the series solutions (§3) even in the non-shallow range of values of the ratio $\mu = l_0/\beta_0$ in figure 1. A summary of all numerical calculations, including those obtained in §5, are tabulated in tables 1–5 of §3. The close agreement in the important case of the lowest antisymmetric mode between the frequencies calculated from the asymptotic and third approximation (series) solutions are particularly noteworthy (see tables 1 and 4).

Finally, it may be noted that, while the developments of this paper are particularly significant when the parameter μ is not necessarily large, the results obtained may also be of interest for seiches over basins of variable depth. Although ordinarily the seiche problem is associated with free-surface oscillations when μ is large (Wehausen & Laitone 1960; Wilson 1972; Csanady 1975), in higher modes of oscillations the frequency becomes progressively larger so that for a given mode it depends on both the parameter μ and the mode number in question. This should be evident from the exact solution of the classical shallow-water theory for a parabolic bottom given by Lamb (1932, p. 277, equation (18)), where the frequency (in our notation) has the form

$$\omega^2\left(\frac{\beta_0}{g}\right) = \frac{n(n+1)}{\mu^2},$$

n being the mode number.

2. Statement of the problem. Differential equations for the linearized theory

Consider a basin of variable depth and suppose that it contains an inviscid, homogeneous, incompressible fluid. Let the basin be referred to a fixed system of rectangular Cartesian coordinates $x_i = (x, y, z)$, with associated orthonormal base vectors $\boldsymbol{e}_i = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, $i = 1, 2, 3$. Choose the origin of the coordinate system at the lowest elevation point of the basin, with the z-axis directed upward (figure 1) and the (x, y)-plane of the coordinate system parallel to the free surface in the undisturbed (equilibrium) state.

We are concerned here with the small-amplitude oscillations of the fluid in the basin under the action of gravity but we neglect the effect of surface tension. Let the bottom surface of the fluid in contact with the basin be specified by the equation

$$\bar{\boldsymbol{p}} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \alpha(x, y)\,\boldsymbol{e}_3, \tag{2.1}$$

and the top free surface of the fluid by

$$\hat{\boldsymbol{p}} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \beta(x, y, t)\,\boldsymbol{e}_3, \tag{2.2}$$

where α is a given function of x, y, but β is unknown and must be determined from the solution of the problem. Further, at any section x = constant (see figure 1), h denotes the equilibrium height,

$$\beta_0 = h_{\max} \tag{2.3}$$

is the value of h at $x = 0$, l_0 is a characteristic length of the free surface in equilibrium state, and $\beta - (\alpha + h)$ represents the amplitude of motion in the present configuration at time t. At the surface (2.2) of the fluid there is only a constant normal pressure p_0 (since surface tension is neglected), while at the bottom surface (2.1) of the fluid in contact with the basin the unknown pressure $\bar{p}$ depends on x, y and t.

We employ the linearized system of differential equations which follows from that derived in the context of a restricted theory of a Cosserat (or a directed) fluid sheet by Green & Naghdi (1976a, 1977). In the interest of clarity, we recall that a Cosserat or a directed surface $\mathscr{C}$ comprises a material surface and a director assigned to every point of the material surface. Let the particles of the material surface of $\mathscr{C}$ be identified with a system of Lagrangian coordinates θ^α ($\alpha = 1, 2$) and let the surface occupied by the material surface in the present configuration of $\mathscr{C}$ at time t be referred to as σ. Let $\boldsymbol{r}$ and $\boldsymbol{d}$ denote the position vector of a typical point of σ and the director at the same point, respectively. Then, with reference to the rectangular Cartesian coordinate system introduced earlier, a motion of the directed surface $\mathscr{C}$ may conveniently be expressed in the form

$$\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \psi\boldsymbol{e}_3, \quad \boldsymbol{d} = \phi\boldsymbol{e}_3, \tag{2.4a, b}$$

where x, y, ψ, ϕ are functions of θ^α, t. The specification of the form (2.4b) can always be made in one configuration even in the context of a more general theory of a directed fluid sheet; but the director will not necessarily remain parallel to $\boldsymbol{e}_3$ throughout the motion. However, the theory used here (Green & Naghdi 1977) restricts the director to remain parallel to a fixed direction for all time. In view of the representations (2.4a, b), the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ and the director velocity $\boldsymbol{w} = \dot{\boldsymbol{d}}$ can be written as

$$\boldsymbol{v} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + \lambda\boldsymbol{e}_3, \quad \boldsymbol{w} = w\boldsymbol{e}_3, \tag{2.5a, b}$$

where

$$u = \dot{x}, \quad y = \dot{y}, \quad \lambda = \dot{\psi}, \quad w = \dot{\phi}, \tag{2.6a, b, c, d}$$

and where a superposed dot denotes the material time derivative holding θ^α fixed.

Green & Naghdi (1976b) have shown that the system of nonlinear differential equations of the restricted theory of a directed fluid sheet (Green & Naghdi 1976a, 1977) can also be derived from the three-dimensional theory by approximating the position vector of the (three-dimensional) continuum in the form

$$\boldsymbol{p}^* = \boldsymbol{r} + \theta^3\phi\boldsymbol{e}_3, \tag{2.7}$$

where θ^3 is a third convected (Lagrangian) coordinate. Corresponding to the values $\theta^3 = \pm\frac{1}{2}$ the expression (2.7) locates the bottom and top surfaces (2.1) and (2.2) of the fluid, respectively. In fact, the quantities ψ and ϕ are related to β and α by (Green & Naghdi 1976b)

$$\psi = \tfrac{1}{2}(\beta + \alpha), \quad \phi = \beta - \alpha. \tag{2.8}$$

In the context of the linearized theory, it can be shown that the velocity components w and λ are related to u, v and their partial derivatives by the expressions

$$w = -h(u_x + v_y), \quad \lambda = \tfrac{1}{2}w - (h_x u + h_y v), \tag{2.9a, b}$$

where $h = \beta_0 - \alpha$ is the equilibrium depth of the water (see figure 1). Then, the relevant system of linearized differential equations for incompressible homogeneous inviscid fluid sheets resulting from the condition of incompressibility and the equations of motion are given by

$$\phi_t = -(hu)_x - (hv)_y, \tag{2.10}$$

$$\rho^* h u_t = -p_x + h_x \bar{p}, \quad \rho^* h v_t = -p_y + h_y \bar{p}, \tag{2.11a, b}$$

$$\rho^* h \lambda_t = \bar{p} - \rho^* g\phi, \quad \tfrac{1}{12}\rho^* h^2 w_t = p - \tfrac{1}{2}h\bar{p} - \tfrac{1}{2}\rho^* g h\phi. \tag{2.12a, b}$$

Some of the symbols in the above equations were defined previously in §1. Others are defined as follows:

ρ^* = the mass density of the fluid.
ϕ = the amplitude of the motion of the free surface of the water.
$\bar{p}$ = pressure in the fluid at its bottom surface.
p = the Lagrange multiplier, i.e. an arbitrary function of position and time.

For later use, we reduce the system of differential equations (2.10)–(2.12) to a system of two partial differential equations in the variables u and v. Thus, introducing (2.9a, b) into (2.12a,b), we obtain two equations for $\bar{p}$ and p which when substituted into

(2.11 a, b) result in two equations involving u, v and ϕ. Next, with the help of (2.10), elimination of ϕ leads to the following two partial differential equations in u and v:

$$[(1-\tfrac{1}{2}hh_{xx})\,u-hh_x\,u_x-\tfrac{1}{3}h^2u_{xx}]_{tt}-g(hu)_{xx} = g(hv)_{xy}+[\tfrac{1}{2}hh_{xy}\,v+\tfrac{1}{2}h(h_x\,v_y+h_y\,v_x)+\tfrac{1}{3}h^2v_{xy}]_{tt}, \quad (2.13a)$$

$$[(1-\tfrac{1}{2}hh_{yy})\,v-hh_y\,v_y-\tfrac{1}{3}h^2v_{yy}]_{tt}-g(hv)_{yy} = g(hu)_{xy}+[\tfrac{1}{2}hh_{xy}\,u+\tfrac{1}{2}h(h_x\,u_y+h_y\,u_x)+\tfrac{1}{3}h^2u_{xy}]_{tt}. \quad (2.13b)$$

The above equations simplify considerably in the special case for which the fluid motion is confined to the (vertical) (x,z)-plane. In this case, since the velocity v in the e_2-direction is zero, all kinematical variables depend on x, t only and (2.13a,b) reduce to the single differential equation†

$$(1-\tfrac{1}{2}hh_{xx})\,u_{tt}-hh_x\,u_{ttx}-\tfrac{1}{3}h^2u_{ttxx} = g(hu)_{xx}. \quad (2.14)$$

The differential equations (2.13a,b), and hence also (2.14), include the effect of vertical inertia within the scope of the approximate theory employed here. This effect is entirely absent in the corresponding classical results as discussed by Lamb (1932, arts 189 and 193). For example, in the special case in which the motion is confined to the (vertical) (x,z)-plane, after the neglect of vertical inertia an equation involving u corresponding to (2.14) is given by

$$u_{tt} = g(hu)_{xx}. \quad (2.15)$$

While the right-hand side of (2.15) is the same as the right-hand side of (2.14) the left-hand side of (2.15) is considerably simpler than the left-hand side of (2.14). In the next section, with the use of (2.14) we examine free-surface oscillations of two symmetric basin profiles with the objective of providing some insight regarding the predictive capability of (2.14) in comparison with that of (2.15).

3. Two-dimensional free-surface oscillations for special basin profiles

Our aim in this section is to provide a quantitative assessment of the nature of the approximate theory characterized by the differential equation (2.14) and hence also by (2.13a,b), as well as comparison of the numerical results with the asymptotic solution of §4. For this purpose, we consider free-surface oscillations in the (x,z)-plane for two symmetric basin profiles, namely (1) a basin with a triangular bottom whose sides of constant slope make an angle θ with the horizontal (see figure 2) and (2) a basin with a parabolic bottom. The solution obtained in the former case, for the two special values of the angle $\theta = \frac{1}{4}\pi$ and $\theta = \frac{1}{6}\pi$, can then be compared with the known exact solutions (Lamb 1932, art. 258) mentioned in §1. Similarly, in the case of a parabolic bottom, comparison is made with a corresponding recently obtained result given by Miles (1985) who has dealt with this problem by an entirely different approach.

Before turning to our objective in this section, we assume a solution of the form

$$u = U(x)\cos\omega t, \quad (3.1)$$

† This differential equation for fluid motions in the (x,z)-plane follows also from one obtained previously by Green & Naghdi (1976b, equation (5.7)), which includes the effect of surface tension.

where the frequency ω is a constant. Then, after substitution of (3.1) in (2.14) we obtain an ordinary differential equation in U given by

$$h(3a-h)\,U_{xx}+3h_x(2a-h)\,U_x+3[1+\tfrac{1}{2}h_{xx}(2a-h)]\,U=0, \tag{3.2}$$

where the constant a is defined by

$$a=\frac{g}{\omega^2}. \tag{3.3}$$

3.1. *A triangular basin*

Consider a symmetric triangular basin whose bottom may be described by (see also figure 2)

$$\alpha(x)=\begin{cases}-(\beta_0/l_0)\,x, & -l_0\leqslant x\leqslant 0,\\ (\beta_0/l_0)\,x, & 0\leqslant x\leqslant l_0,\end{cases} \tag{3.4}$$

where the maximum equilibrium height β_0 is defined by (2.3) and l_0 is now the half length in the undisturbed free surface in the x-direction. Since $\alpha(x)$ is symmetric with respect to the z-axis, in the analysis that follows it will suffice to consider only one-half of the basin. Thus, with $\alpha(x)$ specified by the first of (3.4), the equilibrium height $h(x)$ of the basin, as well as its first two derivatives, are

$$h(x)=\beta_0\left(1+\frac{x}{l_0}\right)=\beta_0(1+\eta),\quad -l_0\leqslant x\leqslant 0, \tag{3.5a}$$

$$h_x=\beta_0/l_0,\quad h_{xx}=0, \tag{3.5b}$$

where for later convenience we have also introduced a dimensionless variable

$$\eta=\frac{x}{l_0}. \tag{3.6}$$

Substitution of (3.5) in (3.2) results in

$$(l_0+x)\,[3a\mu+(l_0+x)]\,U_{xx}+3[2a\mu-(l_0+x)]\,U_x+3\mu^2U=0, \tag{3.7}$$

where we have introduced the parameter

$$\mu=\frac{l_0}{\beta_0}, \tag{3.8}$$

and

$$a\mu=\frac{l_0}{\Omega^2},\quad \Omega=\omega\left(\frac{\beta_0}{g}\right)^{\frac{1}{2}}. \tag{3.9a, b}$$

The parameter μ defined by (3.8) may be regarded as a measure of shallowness of the basin and the constant Ω in (3.9*b*) represents a dimensionless frequency. By a change of the independent variable, namely $\xi=(l_0+x)/(3a\mu)=\frac{1}{3}\Omega^2(1+\eta)$, (3.7) can be reduced to the form

$$\xi(1-\xi)\,U_{\xi\xi}-(2-3\xi)\,U_\xi+3\mu^2U=0,$$

which is a special case of the standard hypergeometric equation (Abramowitz & Stegun 1965, pp. 562–563; compare with entry 15.51 after identifying $a+b=2$, $ab=-3\mu^2$, $c=2$). Of the two linearly independent solutions of the hypergeometric equation only one remains bounded everywhere, and the other (being proportional to ξ^{-1}) becomes unbounded as $\xi\to 0$ (or $x\to -l_0$). Since we require the velocity (and hence U) to remain bounded, the desired solution for u in terms of a hypergeometric function ($F(a, b, c; \xi)$) is given by

$$u(\eta, t) = CF(a, b, c; \tfrac{1}{3}\Omega^2(1+\eta)) \cos \omega t, \tag{3.10}$$

where C is a constant and the temporary notation a, b, c, in the arguments of F, have the values

$$a = 1-(1+3\mu^2)^{\frac{1}{2}}, \quad b = 1+(1+3\mu^2)^{\frac{1}{2}}, \quad c = 2.$$

Then, substitution of (3.10) into (2.10) with $v = 0$ followed by integration, and use of the initial condition $\phi(x, 0) = 0$ (for motions which begin from rest) leads to

$$\phi(\eta, t) = -\frac{C}{\omega\mu}\{(1+\eta) F(a, c, c; \tfrac{1}{3}\Omega^2(1+\eta))\}_{\eta} \sin \omega t. \tag{3.11}$$

The results (3.10) and (3.11) represent the solution of the problem under consideration. The frequency equation associated with the antisymmetric and symmetric modes of oscillations are obtained from the conditions

$$\phi(0, t) = 0, \quad u(0, t) = 0, \tag{3.12a, b}$$

respectively. We note that the hypergeometric function F in (3.10) and (3.11) admits a series representation which, for our present purpose, may be written as

$$F(a, b, c; \tfrac{1}{3}\Omega^2(1+\eta)) = 1+c_1(1+\eta)+c_2(1+\eta)^2+c_3(1+\eta)^3+\ldots, \tag{3.13a}$$

$$c_1 = -\tfrac{1}{2}\mu^2\Omega^2, \quad c_2 = \tfrac{1}{12}\mu^2(\mu^2-1)\,\Omega^4, \quad c_3 = -\tfrac{1}{432}\mu^2(\mu^2-1)(3\mu^2-8)\,\Omega^6. \tag{3.13b}$$

In order to obtain the natural frequencies associated with the lowest modes of oscillations of water in the basin, we use the series representation (3.13a) and write (3.10) and (3.11) in the form

$$u(\eta, t) = C\{1+c_1(1+\eta)+c_2(1+\eta)^2+c_3(1+\eta)^3+\ldots\} \cos \omega t, \tag{3.14a}$$

$$\phi(\eta, t) = -\frac{C}{\omega\mu}\{[1+c_1(1+\eta)+c_2(1+\eta)^2+\ldots] + (1+\eta)[c_1+2c_2(1+\eta)+\ldots]+\ldots\} \sin \omega t. \tag{3.14b}$$

Now, adopting a usual procedure, for small oscillations we consider the predictions of (3.14) to the first few (more specifically three) orders of approximation in Ω^2 and calculate the frequencies in accordance with the conditions (3.12a, b). In this way, for the first approximation, (3.12a) and (3.12b) give respectively $1+2c_1 = 0$ and $1+c_1 = 0$; and these, in turn, yield the dimensionless frequencies

$$\Omega_{1\mathrm{a}} = \frac{1}{\mu}, \quad \Omega_{1\mathrm{s}} = \frac{2^{\frac{1}{2}}}{\mu}, \tag{3.15a, b}$$

where the subscript 1 attached to Ω signifies the first approximation of the series solution (3.14a, b) and the subscripts a and s refer to the antisymmetric and symmetric modes, respectively. The frequencies (3.15a, b) are the same as those predicted by the classical developments in which the effect of vertical inertia is neglected. In particular, they may be obtained (after an adjustment in notation) from the leading terms of a solution given by Lamb (1932, p. 276, equation (7)) and they can also be deduced directly from the first approximation of a series solution of (2.15).

For the second and third approximations, the respective frequency equations which follow from the conditions (3.12a, b) are more intricate and their roots in each case must be calculated numerically. These frequency equations, in the case of second approximations, are given by:

$$\mu^2(\mu^2-1)\,\Omega_{2a}^4-4\mu^2\Omega_{2a}^2+4=0 \quad \text{(antisymmetric mode)}, \tag{3.16a}$$

$$\mu^2(\mu^2-1)\,\Omega_{2s}^4-6\mu^2\Omega_{2s}^2+12=0 \quad \text{(symmetric mode)}. \tag{3.16b}$$

Similarly, in the case of the third approximation, we have:

$$\mu^2(\mu^2-1)(3\mu^2-8)\,\Omega_{3a}^6-27\mu^2(\mu^2-1)\,\Omega_{3a}^4+108\mu^2\Omega_{3a}^2-108=0, \quad \text{(antisymmetric mode)}, \tag{3.17a}$$

$$\mu^2(\mu^2-1)(3\mu^2-8)\,\Omega_{3s}^6-36\mu^2(\mu^2-1)\,\Omega_{3s}^4+216\mu^2\Omega_{3s}^2-432=0 \quad \text{(symmetric mode)}. \tag{3.17b}$$

In the expressions (3.16)–(3.17), the subscripts 2 and 3 attached to Ω signify, respectively, the second and third approximations in the series solution; and again the subscripts a and s refer, respectively, to the antisymmetric and symmetric modes.

Before considering detailed numerical comparisons of the lowest roots of (3.16*a*, *b*) and (3.17*a*, *b*) with the results (3.15*a*, *b*) of the first approximation, it is of interest to discuss predictions of (3.15*a*, *b*) in the two special cases (corresponding to the values $\frac{1}{4}\pi$ and $\frac{1}{6}\pi$ of the angle θ in figure 2) for which *exact* solutions are available (Lamb 1932, pp. 443–444):

$$\text{For } \mu = 1\ (\theta = 45^\circ), \quad \Omega_a = 1.000, \quad \Omega_s = 1.5244 \quad \text{(Kirchhoff 1879)} \tag{3.18a, b}$$

$$\text{For } \mu = 3^{\frac{1}{2}}\ (\theta = 30^\circ), \quad \Omega_s = 1.000 \quad \text{(Greenhill 1887)}. \tag{3.19}$$

Although Greenhill attempted to discuss both the symmetric and the antisymmetric modes, his solution (Greenhill 1887) is limited to the symmetric mode only; and, in the 6th edition of his book, Lamb (1932, p. 444) correctly remarks that the frequency for the antisymmetric mode when $\mu = 3^{\frac{1}{2}}$ 'has not yet been determined'. Since then several related papers by Sen (1927) and Storchi (1949, 1952) have dealt with the exact solutions of special basin profiles but none of these appear to contain a solution for the antisymmetric mode of a triangular basin when $\theta = \frac{1}{6}\pi$.

The predictions of the first approximation given by (3.15*a*, *b*) for $\mu = 1$ when compared with the exact solutions (3.18*a*, *b*) is better than one would expect from a relatively simple theory utilized in the present paper. For $\mu = 1$, (3.15*a*) for the antisymmetric mode gives the same value 1.000 as the exact result (3.18*a*) while the prediction of (3.15*b*) for the symmetric mode ($\Omega_s = 1.414$) is about 7.2 % less than the exact value (3.18*b*). Also, for $\mu = 3^{\frac{1}{2}}$, the prediction of (3.15*b*) for the symmetric mode ($\Omega_s = 0.816$) is about 18.4 % below that of the exact value in (3.19). However, as will be noted below, the predictions of the second and third approximations of the series solution (3.14) in the symmetric mode are much closer to the exact value (3.19).

The unusually good agreement in the case of the antisymmetric mode for $\mu = 1$ requires some explanation. Indeed, an examination of the coefficients (3.13*b*) of the series representation of the hypergeometric function in (3.13*a*) indicates that the series terminates for $\mu = 1$ beyond the first approximation so that the first approximation in (3.14) for $\mu = 1$ is the exact solution of the differential equation (3.7). Since the hypergeometric series terminates for $\mu = 1$, consideration of higher approximations in this case will not yield any new result for either of the two modes of oscillations.

In fact, these conclusions or the case of $\mu = 1$ hold also when the effect of vertical inertia is entirely neglected as in the classical developments. (See also the remarks following (3.15*a*, *b*).)

Each of the frequency equations (3.16*a*, *b*) and (3.17*a*, *b*) for $\mu = 1$ has only one positive root and yields the value of the first approximation (3.15*a*, *b*) in line with the remarks made in the preceding paragraph. For values of $\mu \neq 1$, the character of the roots of (3.16*a*, *b*) and (3.17*a*, *b*) may be summarized as follows:

$$(3.16a):\quad \begin{cases} \text{For } 0 < \mu < 1, & \text{one real } \Omega > 0, \\ \text{for } 1 < \mu, & \text{two real } \Omega > 0. \end{cases}$$

$$(3.16b):\quad \begin{cases} \text{For } 0 < \mu < 1, & \text{one real } \Omega > 0, \\ \text{for } 1 < \mu < 2, & \text{two real } \Omega > 0, \\ \text{for } 2 < \mu, & \text{no real } \Omega. \end{cases}$$

$$(3.17a):\quad \begin{cases} \text{For } 0 < \mu < 1 \text{ and } (\simeq 2.4) < \mu, & \text{one real } \Omega^2, \\ \text{for } 1 < \mu < (\simeq 2.3), & \text{three real unequal } \Omega^2. \end{cases}$$

$$(3.17b):\quad \begin{cases} \text{For } 0 < \mu < 1 \text{ and } (\simeq 2.3) < \mu, & \text{one real } \Omega^2, \\ \text{for } 1 < \mu < (\simeq 2.2), & \text{three real unequal } \Omega^2. \end{cases}$$

Whenever the frequency equation has more than one real root, we have recorded only the lowest root in the tabulated results of tables 1–3 (and also later in tables 4–5).

Numerical values of the lowest roots in the second and third approximation of the series solution (3.14) as well as those of the first approximation (classical results with vertical inertia neglected), are tabulated for both the lowest antisymmetric and the lowest symmetric modes in tables 1 and 2, respectively. To avoid repetition of a display of these calculated results and for the reader's later convenience, we have also recorded in tables 1 and 2 the corresponding values of the frequencies calculated in §5 from the asymptotic solution of §4. As was noted earlier, there is no real root for $\mu > 2$ in the symmetric mode of the second approximation (see also the fourth column of table 2). Moreover, as indicated in table 3, for $\mu > 3^{\frac{1}{2}}$, Ω_{2s} takes a value less than 1.000 and continues to decrease until $\mu \simeq 1.96$ and then begins to increase until it assumes again the value 1.000 for $\mu = 2$, beyond which there is no real root. This should explain why the values recorded in table 2 for both $\mu = 3^{\frac{1}{2}}$, 2 are 1.000.

For values of $\mu \geqslant 5$ the results in the antisymmetric mode obtained from the asymptotic solution (Ω_{Aa}) and the third series approximation ($\Omega_{3\mathrm{a}}$) are very close. In fact, the percentage difference between the values of Ω_{Aa} and $\Omega_{3\mathrm{a}}$ over the range $3^{\frac{1}{2}} \leqslant \mu < 10$ is less than 5%. For values larger than $\mu > 10$ the different approximations all approach the values predicted by the first approximation (3.15), which is the same as the classical result.

3.2. *A parabolic basin*

Consider a symmetric parabolic basin and specify the one-half of its depth profile by

$$h(x) = \beta_0\left(1-\frac{x^2}{l_0^2}\right) = \beta_0(1-\eta^2), \quad -l_0 \leqslant x \leqslant 0, \tag{3.20a}$$

$$h_\eta = -2\beta_0\eta, \quad h_{\eta\eta} = -2\beta_0, \tag{3.20b}$$

where η is defined by (3.6). Substitution of (3.20) in (3.2) leads to

$$(1-\eta^2)\left[\frac{3}{\Omega^2}-(1-\eta^2)\right]U_{\eta\eta}-6\eta\left[\frac{2}{\Omega^2}-(1-\eta^2)\right]U_\eta+3\left[\mu^2-\frac{2}{\Omega^2}+(1-\eta^2)\right]U = 0. \tag{3.21}$$

$\mu = \dfrac{l_0}{\beta_0}$	θ in degrees	Ω_{1a}	Ω_{2a}	Ω_{3a}	Ω_{Aa}
1.0	45.000	1.000	1.000	1.000	1.142
$3^{\frac{1}{2}}$	30.000	0.577	0.650	0.648	0.682
2.0	26.565	0.500	0.577	0.571	0.593
3.0	18.435	0.333	0.408	0.391	0.398
4.0	14.036	0.250	0.316	0.296	0.300
5.0	11.310	0.200	0.258	0.238	0.240
10.0	5.711	0.100	0.135	0.119	0.120
20.0	2.862	0.050	0.069	0.060	0.060

TABLE 1. Comparison of the numerical values of the non-dimensional frequency in the antisymmetric mode for a triangular basin of constant slope over the range $1 \leqslant \mu \leqslant 20$ from various approximations: the frequencies Ω_{1a}, Ω_{2a}, Ω_{3a} calculated from the first three approximations in the series solution (3.14) and Ω_{Aa} calculated from the asymptotic solution (4.20). For each entry in the table, the value of angle θ (defined in figure 2) is also indicated.

$\mu = \dfrac{l_0}{\beta_0}$	θ in degrees	Ω_{1s}	Ω_{2s}	Ω_{3s}	Ω_{As}
1.0	45.000	1.414	1.414	1.414	1.548
$3^{\frac{1}{2}}$	30.000	0.816	1.000	0.987	1.032
2.0	26.565	0.707	1.000	0.879	0.910
3.0	18.435	0.471	—	0.606	0.624
4.0	14.036	0.354	—	0.457	0.473
5.0	11.310	0.283	—	0.366	0.380
10.0	5.711	0.141	—	0.183	0.191
20.0	2.862	0.071	—	0.091	0.096

TABLE 2. Comparison of the numerical values of the non-dimensional frequency in the symmetric mode for a triangular basin of constant slope over the range $1 \leqslant \mu \leqslant 20$ from various approximations: the frequencies Ω_{1s}, Ω_{2s}, Ω_{3s} calculated from the first three approximations in the series solution (3.14) and Ω_{As} calculated from the asymptotic solution (4.20). For each entry in the table, the value of angle θ (defined in figure 2) is also indicated.

By a change of the independent variable, namely $\xi = \eta + 1$, we seek a series solution of (3.21) in the form

$$U(\xi) = \sum_{n=0}^{\infty} c_n \xi^{r+n},$$

valid about the singular point $\eta = -1$. By the usual procedure, after substituting the above series in (3.21) and making some rearrangements, we put the coefficient of the lowest power of ξ equal to zero and obtain the indicial equation $r(r+1) = 0$. Here we only need to consider the value $r = 0$ since the other value $r = -1$ gives rise to an unbounded solution. Thus, after setting the coefficient $c_0 = 1$, we finally obtain the expressions

$$u(\eta, t) = M\{1 + c_1(1+\eta) + c_2(1+\eta)^2 + c_3(1+\eta)^3 + \ldots\} \cos \omega t, \tag{3.22a}$$

$$\phi(\eta, t) = -\frac{M}{\omega\mu}\{-2\eta[1 + c_1(1+\eta) + c_2(1+\eta)^2 + c_3(1+\eta)^3 + \ldots] + (1-\eta^2)[c_1 + 2c_2(1+\eta) + 3c_3(1+\eta)^2 + \ldots]\} \sin \omega t, \tag{3.22b}$$

$\mu = \dfrac{l_0}{\beta_0}$	Ω_{2s}
$^{-}3^{\frac{1}{2}}$	1.000
1.76	0.992
1.80	0.982
1.84	0.974
1.88	0.967
1.92	0.964
1.96	0.965
2.00	1.000

TABLE 3. Extended calculated results for the frequency Ω_{2s} of the second approximation in table 2 over the range $1 \leqslant \mu \leqslant 2$

$\mu = \dfrac{l_0}{\beta_0}$	Ω_{1a}	Ω_{2a}	Ω_{3a}	Ω_{Aa}	Ω_{Ma}
1.0	1.141	1.155	1.070	1.185	1.054
1.5	0.943	0.837	0.809	0.858	0.810
2.0	0.707	0.655	0.644	0.663	0.645
3.0	0.471	0.453	0.451	0.452	0.452
4.0	0.354	0.345	0.345	0.341	0.345
5.0	0.283	0.279	0.278	0.274	0.278
10.0	0.141	0.141	0.141	0.138	0.141
20.0	0.071	0.071	0.071	0.069	0.071

TABLE 4. Comparison of the numerical values of the non-dimensional frequency in the antisymmetrical mode for a parabolic basin over the range $1 \leqslant \mu \leqslant 20$ from various approximations: the frequencies Ω_{1a}, Ω_{2a}, Ω_{3a} calculated from the first three approximations in the series solution (3.22), Ω_{Aa} calculated from the asymptotic solution (4.20) and Ω_{Ma} calculated from the variational approximation (3.27) of Miles (1985)

where M is a constant and the coefficients c_1, c_2, c_3, ..., are:

$$\left.\begin{aligned} c_1 &= \frac{2/\Omega^2-\mu^2}{4/\Omega^2}, \quad c_2 = \frac{12/\Omega^4-8\mu^2/\Omega^2+\mu^2(\mu^2-4)}{48/\Omega^4}, \\ c_3 &= \frac{[36/\Omega^2+32-3\mu^2]\,c_2-24c_1+3}{72/\Omega^2}. \end{aligned}\right\} \tag{3.23}$$

It should be noted that the expression for the wave height ϕ in (3.22*b*) is obtained similarly to (3.11), after substitution of (3.22*a*) into (2.10) with $v = 0$ followed by integration and the use of the initial condition $\phi(x, 0) = 0$.

For small oscillations, we again consider the first approximation in the series solution, retaining terms which involve only c_1 in (3.22). For the lowest modes of oscillations, the natural frequencies of the first approximation, calculated in accordance with the conditions (3.12*a, b*), lead to the solution given by Lamb (1932, p. 277, equation (18) with $n = 1$ and $n = 2$), i.e.

$$\Omega_{1a} = \frac{2^{\frac{1}{2}}}{\mu}, \quad \Omega_{1s} = \frac{6^{\frac{1}{2}}}{\mu}. \tag{3.24a, b}$$

$\mu = \dfrac{l_0}{\beta_0}$	Ω_{1s}	Ω_{2s}	Ω_{3s}	Ω_{As}
1.0	2.449	1.709	1.504	1.599
1.5	1.633	1.278	1.193	1.322
2.0	1.225	1.025	0.991	1.082
3.0	0.816	0.734	0.734	0.770
4.0	0.612	0.571	0.576	0.591
5.0	0.490	0.467	0.471	0.478
10.0	0.245	0.241	0.243	0.243
20.0	0.122	0.122	0.122	0.122

TABLE 5. Comparison of the numerical values of non-dimensional frequency in the symmetric mode for a parabolic basin over the range $1 \leqslant \mu \leqslant 20$ from various approximations: the frequencies Ω_{1s}, Ω_{2s}, Ω_{3s} calculated from the first three approximations in the series solution (3.22) and Ω_{As} calculated from the asymptotic solution (4.20)

For the second and third approximations, the respective frequency equations which follow from the conditions (3.12*a*, *b*) are more intricate and their roots in each case must be calculated numerically. In the case of the second approximation, the frequency equations are given by:

$$\mu^2(\mu^2-4)\,\Omega_{2a}^4-14\mu^2\Omega_{2a}^2+24 = 0 \quad \text{(antisymmetric mode)}, \tag{3.25a}$$

$$\mu^2(\mu^2-4)\,\Omega_{2s}^4-20\mu^2\Omega_{2s}^2+84 = 0 \quad \text{(symmetric mode)}. \tag{3.25b}$$

Similarly, in the case of the third approximation, we have:

$$\mu^2(3\mu^4-44\mu^2+128)\,\Omega_{3a}^6-\mu^2(108\mu^2-304)\,\Omega_{3a}^4+(996\mu^2+48)\,\Omega_{3a}^2-1584 = 0 \quad \text{(antisymmetric mode)}, \tag{3.26a}$$

$$\mu^2(3\mu^4-44\mu^2+128)\,\Omega_{3s}^6-\mu^2(132\mu^2-400)\,\Omega_{3s}^4+(1764\mu^2+48)\,\Omega_{3s}^2-6480 = 0 \quad \text{(symmetric mode)}. \tag{3.26b}$$

Again, as in the discussion of §3.1 the subscripts 1, 2, 3 attached to Ω signify, respectively, the first, second and third approximations in the series solution; and the subscripts a and s refer, respectively, to the antisymmetric and symmetric modes. Whenever the frequency equation has more than one real root, we have recorded only the lowest root in the tabulated results of tables 4 and 5.

Before considering a comparison of the numerical values of the various approximations, mention should be made of a recent paper by Miles which also briefly deals with the free-surface oscillation of a parabolic basin using a variational approximation (Miles 1985, paragraph following his equation (4.5*b*)). His frequency for the antisymmetric mode in our notation reads as (the symbols d, a, δ in the paper of Miles correspond, respectively, to β_0, l_0, $1/\mu$ used here):

$$\Omega_{Ma} = 2^{\frac{1}{2}}(\mu^2+\tfrac{4}{5})^{-\frac{1}{2}}. \tag{3.27}$$

Numerical values of the lowest roots of the frequency equations in the second and third approximation of the series solution (3.22), as well as the frequencies of the first approximation (3.24*a*, *b*),† are tabulated for both the lowest antisymmetric and the

† Recall that these are the same as the classical results with vertical inertia neglected (Lamb 1932, p. 277, equation (18) with $n = 1$ and $n = 2$).

lowest symmetric modes in tables 4 and 5, respectively. To avoid repetition of a display of these calculated results and for the reader's later convenience, we have also included in tables 4 and 5 the corresponding values of the frequencies calculated in §5 from the asymptotic solution of §4. In addition, in the case of table 4, we have recorded the prediction of the frequencies from the variational approximation of Miles (1985).

For values of $\mu \geqslant 3$, the results in the antisymmetric mode obtained from the asymptotic solution (Ω_{Aa}), the variational approximation of Miles (Ω_{Ma}) and the series approximation ($\Omega_{3\mathrm{a}}$) are very close. In fact, the absolute value of the percentage difference between the values of Ω_{Aa} and $\Omega_{3\mathrm{a}}$ (which is also very nearly the same as Ω_{Ma}) is less than 3% over the range $2 \leqslant \mu \leqslant 20$. For values larger than $\mu \geqslant 20$ the different approximations all approach the values predicted by the first approximation (3.24), which is the same as the classical result. The comparative accuracy in the symmetric mode between Ω_{As} and $\Omega_{3\mathrm{s}}$ is about the same: the absolute value of the percentage difference between Ω_{As} and $\Omega_{3\mathrm{s}}$ is less than 5% for $\mu = 3$, the percentage difference decreases for larger values of μ, and is less than 3% for $3.8 \leqslant \mu \leqslant 20$.

4. Solution of differential equations (2.13) by asymptotic integration

In this section, we confine attention to a class of basin profiles whose equilibrium depth varies with x only, and obtain a solution of the differential equations (2.13) by asymptotic integration. We specify a class of basin profiles by

$$h(\eta) = (\eta-\eta_0)\bar{h}(\eta), \quad \bar{h}(\eta_0) \neq 0, \tag{4.1}$$

where η is defined by (3.6), l_0 is again a characteristic length in the free surface of water such as that indicated in figure 1 and η_0 is the value of η at which h vanishes such as the point $x = -l_0$ (or $\eta = -1$) in figure 1. It should be noted here that the form (4.1) covers a fairly large class of basin profiles. For example, the expression (4.1) includes any profile of the form

$$h(\eta) = \eta^n - c^n = (\eta-c)[\eta^{n-1}+c\eta^{n-2}+c^2\eta^{n-3}+\ldots+c^{n-2}\eta+c^{n-1}], \tag{4.2}$$

where c is a constant.

Assuming a solution which is sinusoidal in time and periodic in the y-direction, we write

$$u = U(x)\cos ky \cos\omega t, \quad v = V(x)\sin ky\cos\omega t, \tag{4.3a,b}$$

where both the wavenumber k and the frequency ω are constants. Thus, with h specified in the form (4.1), substitution of (4.3a,b) into the system of differential equations (2.13a,b) leads to the following two coupled ordinary differential equations U and V:

$$-[k^2h(h-3a)+3]\,V = \tfrac{3}{2}kh_x(h-2a)\,U + kh(h-3a)\,U_x, \tag{4.4}$$

$$h(h-3a)\,U_{xx} + 3h_x(h-2a)\,U_x + \tfrac{3}{2}[h_{xx}(h-2a)-2]\,U = -\tfrac{3}{2}kh_x(h-2a)\,V - kh(h-3a)\,V_x, \tag{4.5}$$

where the constant a is defined by (3.3).

It is convenient to eliminate V from the above system of equations, thereby obtaining a second-order differential equation in U. Then, (4.4) provides an expression for V in terms of U and its derivative U_x. In this manner, after introducing a change of variable for U defined by

$$r(\eta) = h\left\{\frac{3-k^2h(3a-h)}{3a-h}\right\}^{-\frac{1}{2}} U(x), \tag{4.6}$$

the coupled system of equations (5.4)–(5.5) can be reduced to

$$r_{\eta\eta}+\{\mu^2(\eta-\eta_0)^{-1}\psi_1(\eta)+\tau(\eta)\}r=0. \tag{4.7}$$

In (4.7) the parameter μ is again the ratio of a characteristic length l_0 to the maximum equilibrium depth $h_{\max}$ of the basin, the coefficient functions ψ_1 and τ are given by†

$$\psi_1(\eta)=\frac{h_{\max}(3-k^2h(3a-h))}{\bar{h}(\eta)(3a-h)}, \tag{4.8}$$

$$\begin{aligned}\tau(\eta)=\tfrac{1}{2}h_{\eta\eta}\frac{F'}{F}+h_\eta^2\bigg\{&\frac{3}{4}\frac{4a-h}{h(3a-h)^2}-\frac{3}{4}\left(\frac{F'}{F}\right)^2+\frac{k^2}{F}-\frac{k^2}{2}\\&+\tfrac{3}{4}k^2\frac{(2a-h)^2}{h(3a-h)}-\frac{F'}{F}\left(\frac{1}{2(3a-h)}-\frac{1}{h}+\frac{k^2(2a-h)}{2}\right)\bigg\},\end{aligned} \tag{4.9}$$

and where

$$F(h,a,k)=3-k^2h(3a-h),\quad F'(h,a,k)=\frac{\partial F}{\partial h}=-k^2(3a-2h). \tag{4.10}$$

It should be noted also that, in obtaining (4.7), the change of variable (4.6) is derived under the condition that the resulting differential equation in r (as the dependent variable) would not contain the first-order derivative of r. In the process of deriving (4.7) two possibilities present themselves for the sign of the quantity $(3a-h)$, which occurs in both the numerator and denominator of (4.6) and in the solution (4.11) given below. Here we have specified the sign of $(3a-h)$ to be positive so that $3a>h$. The other choice, namely $3a-h<0$ or equivalently $\omega^2>3g/h$, is discarded on physical grounds: it gives undesirable restrictions in that it allows for the possibility of $\omega^2>\infty$ as $h\to 0$ and also requires that ω^2 can never be less than a positive number $3g/h$ for non-zero values of h.

We now turn to a discussion of a solution of (4.7) by a method of asymptotic integration which is valid at the turning point of ordinary differential equations. For a fairly general account of this procedure reference may be made to a recent book on the subject by Meyer & Parter (1980). However, for a limited application of this procedure to the second-order differential equation (4.7) it is more convenient to refer directly to a paper by Langer (1935). With reference to (4.7), we note that the function $\psi_1(\eta)$ is a bounded single-valued analytic function and that the coefficient function $\tau(\eta)$ is an analytic function (with respect to η) having a pole of order one at $\eta=\eta_0$ and is, of course, bounded with respect to μ (actually τ is independent of μ).

According to a theorem of Langer (1935)‡ there exists a related differential equation whose solution is asymptotic to the solution of (4.7) and its domain of validity is dependent upon the character of the coefficient functions of r in (4.7). If η ranges over an interval which includes the point η_0 at which (i) $\tau(\eta)$ admits a pole of first order, (ii) ψ_1 is analytic and bounded, then the asymptotic solution mentioned will be valid in the entire interval under consideration including η_0.

It can then be shown that the two linearly independent solutions of (4.7) are given by (the details are similar to the corresponding development in Naghdi 1957):

$$\begin{Bmatrix}r_1\\r_2\end{Bmatrix}=\begin{Bmatrix}C_1\\C_2\end{Bmatrix}\left(\frac{1}{h_{\max}}\right)^{\frac{1}{2}}\left[\frac{(3a-h)h}{3-k^2h(3a-h)}\right]^{\frac{1}{4}}\xi^{\frac{1}{2}}\begin{Bmatrix}J_1(\xi)\\Y_1(\xi)\end{Bmatrix}, \tag{4.11}$$

† Here and elsewhere in §4, for emphasis we use $h_{\max}$ to designate the equilibrium depth at the lowest elevation point of the basin instead of β_0 defined by (2.3).

‡ A statement of the details of Langer's theorem is summarized in a paper by Naghdi (1957, pp. 50–51).

where

$$\xi = \mu \int_{\eta_0}^{\eta} (\gamma-\eta_0)^{-\frac{1}{2}} \psi_1^{\frac{1}{2}}(\gamma)\,\mathrm{d}\gamma = (\mu h_{\max}) \int_{\eta_0}^{\eta} \left\{\frac{3}{h(\gamma)[3a-h(\gamma)]} - k^2\right\}^{\frac{1}{2}} \mathrm{d}\gamma, \tag{4.12}$$

C_1, C_2 are arbitrary constants of integration and J_1, Y_1 are Bessel functions of the first and second kind of order one. It should be noted here that the correction terms to the above asymptotic solutions are of the order μ^{-2} relative to the solution (4.11); see, in this connection, Langer (1935, p. 407).

Since we require the velocities and the amplitude to be finite throughout the domain under consideration, we discard the solution r_2, which becomes unbounded as $\eta \to \eta_0$ corresponding to $h \to 0$, $\xi \to 0$. Then, recalling (4.6), the asymptotic solution for U is

$$U(x) = M\left[\frac{3-(3A-1)K^2}{(3A-1)h^4}\right]^{\frac{1}{4}} \xi^{\frac{1}{2}} J_1(\xi), \tag{4.13}$$

where we have introduced the notations

$$K = kh, \quad A = \frac{g}{\omega^2 h} = \frac{a}{h}, \tag{4.14a,b}$$

$$M = C_1\left(\frac{1}{h_{\max}}\right)^{\frac{1}{2}} = \text{const.} \tag{4.15}$$

and C_1 or equivalently M in (4.15) is a constant. Having obtained the result (4.13), the expression for V can be readily calculated from (4.4) and is given by

$$V(x) = \frac{K}{2[3-K^2(3A-1)]}[3(2A-1)h_x U + 2(3A-1)hU_x]. \tag{4.16}$$

For a complete discussion of free oscillations of the body of water in the basin, the foregoing solution must be supplemented by appropriate boundary and initial conditions. One of the boundary conditions (pertaining to boundedness of the velocity at $\eta = \eta_0$) has already been utilized and the initial conditions may be specified by either of the following conditions:

$$u(x,y,0) = u_0(x,y) \quad \text{or} \quad \phi(x,y,0) = \phi_0(x,y). \tag{4.17}$$

For our present purpose, we set $\phi_0 = 0$ but impose no initial conditions on the velocity field. This would correspond to the type of oscillations described by Wilson (1972, p. 80) such as the impact of the wind gusts on the undisturbed free surface of water.

With velocities u and v now determined, the solution for the amplitude ϕ of the free surface can be easily determined from (2.10). We record below the final results:

$$u = M\hat{U}(x)\cos ky \cos\omega t, \quad v = M\hat{V}(x)\sin ky \cos\omega t, \tag{4.18a,b}$$

$$\phi = -\frac{M}{\omega}\{[h\hat{U}(x)]_x + K\hat{V}(x)\}\cos ky \sin\omega t, \tag{4.18c}$$

where the functions $\hat{U}$ and $\hat{V}$ are given by

$$\hat{U} = \left[\frac{3-(3A-1)K^2}{(3A-1)h^4}\right]^{\frac{1}{4}} \xi^{\frac{1}{2}} J_1(\xi), \tag{4.19a}$$

$$\hat{V} = \tfrac{1}{2}K[3-K^2(3A-1)]^{-1}\{3(2A-1)h_x\hat{U} + 2(3A-1)h\hat{U}_x\}, \tag{4.19b}$$

and where in obtaining (4.18c) we have imposed the initial condition $\phi = 0$. The special case of the above solution for a two-dimensional motion in which $v = 0$ and

u, ϕ depend on x, t only can be obtained by setting $k = 0$. Then, $K = 0$ by (4.14a) and (4.18a, b, c) reduce to

$$u = M\hat{U}(x)\cos\omega t, \quad v = 0, \quad \phi = -\frac{M}{\omega}[h\hat{U}(x)]_x \sin\omega t, \qquad (4.20a, b, c)$$

where the function $\hat{U}$ is now given by

$$\hat{U} = \left[\frac{3}{(3A-1)h^4}\right]^{\frac{1}{4}} \xi^{\frac{1}{2}} J_1(\xi). \qquad (4.21)$$

It is of interest to consider the limiting values of the solution obtained in this section when $\eta \to \eta_0$. We discuss this with reference to the special solution (4.20) but similar limiting values can be obtained from the more general results (4.18). Now the factor $(\eta - \eta_0)$, which occurs in h and $\xi^{\frac{1}{2}}J_1(\xi)$ (see (4.1) and (4.12)), is present in both the numerator and denominator of (4.21) and results in indeterminacy of the limiting values of u, h as $\eta \to \eta_0$. However, the limiting values can be obtained by making use of the mean-value theorem for integrals. Thus, with the help of (4.12), the recurrence relations for Bessel functions and the limiting values of the Bessel functions as $\xi \to 0$, we obtain the following limiting values of (4.20a, c):

$$\lim_{\eta \to \eta_0} u(\eta, t) = M(\tfrac{4}{3})^{\frac{1}{4}} \left(\frac{\omega^2}{g}\right) \left(\frac{l_0}{h_\eta(\eta_0)}\right)^{\frac{3}{2}} \cos\omega t, \qquad (4.22a)$$

$$\lim_{\eta \to \eta_0} \phi(\eta, t) = -M(\tfrac{4}{3})^{\frac{1}{4}} \left(\frac{\omega}{g}\right) \left(\frac{l_0}{h_\eta(\eta_0)}\right)^{\frac{1}{2}} \sin\omega t. \qquad (4.22b)$$

Before closing this section, we elaborate on the implication of the change of variable (4.6), where we have assumed that the quantity $(3A-1)$ is strictly positive. This, along with an examination of (4.19a, b), suggests that for real and bounded values of the functions $\hat{U}(x)$ and $\hat{V}(x)$ in (4.18) we must require that, for all x,

$$3A - 1 > 0 \quad \text{and} \quad 3 - K^2(3A-1) > 0. \qquad (4.23)$$

The restrictions (4.23), together with the definitions (4.14a, b), imply

$$3A > 1 \Rightarrow \omega^2 < \frac{3g}{h}, \qquad (4.24)$$

$$K^2 < \frac{3}{3A-1} \Rightarrow k^2 < \frac{3\omega^2}{3gh - \omega^2 h^2}, \qquad (4.25)$$

which hold for all x (or η). The quantity $3g/h$ which occurs in (4.24) attains its minimum value when h is a maximum and this corresponds to the value of the cutoff frequency (1.10). Corresponding to this cutoff frequency, it can be shown that the restriction (4.25) provides a lower bound for the constant wavelength λ. More specifically, for $\omega^2 = \frac{1}{2}\omega_c^2$, a lower bound for λ^2 is found to be $\frac{4}{3}\pi^2 h_{\max}^2$. It is interesting that this value is within the same range as that indicated for basins of uniform depth in §1 (compare with (1.7)).

5. Concluding remarks

We close this paper with some numerical examples and remarks pertaining to applications of the general solution discussed in §4.

5.1. *Two examples of the asymptotic solution* (4.20)

We consider here the predictions of the asymptotic solution (4.20) for motions in the (x, z)-plane of two basins whose bottom profiles are (i) triangular and (ii) parabolic. For each of the two basin profiles, we calculate the lowest frequencies in both the antisymmetric and symmetric modes of oscillations. Before carrying out the specific calculations, however, it is desirable to obtain general expressions of the frequency equations from the solution (4.20a, c) with the basin's profile left unspecified. To this end, we first rewrite the argument ξ of the Bessel function in (4.21). Thus, using again the notation (2.3) and since now $k = 0$ in (4.12), we have

$$\xi = \mu\beta_0 \int_{\eta_0}^{\eta} \left\{\frac{3}{h(\gamma)[3a-h(\gamma)]}\right\}^{\frac{1}{2}} \mathrm{d}\gamma, \tag{5.1}$$

together with

$$\xi_\eta = \mu\beta_0 \left[\frac{3}{h(3a-h)}\right]^{\frac{1}{2}}. \tag{5.2}$$

Then, with the use of (4.20c), the frequency equation for the antisymmetric mode obtained from the condition (3.12a) is found to be

$$\left\{\frac{3a}{4}\frac{h_\eta}{h(3a-h)}\xi^{\frac{1}{2}}J_1(\xi)+\xi^{\frac{1}{2}}\xi_\eta J_0(\xi)-\tfrac{1}{2}\xi^{-\frac{1}{2}}\xi_\eta J_1(\xi)\right\}_{\eta=0} = 0 \quad \text{(antisymmetric mode).} \tag{5.3}$$

The condition (3.12b) for the symmetric mode requires that $\hat{U}$ given by (4.21) vanish at $\eta = 0$ or equivalently $\xi^{\frac{1}{2}}J_1(\xi) = 0$ at $\eta = 0$. Recalling (4.12), it can be easily verified that $\xi \neq 0$ at $\eta = 0$. Hence, the condition (3.12b) implies that

$$J_1(\xi)|_{\eta=0} = 0 \quad \text{(symmetric mode).} \tag{5.4}$$

Consider now a triangular basin specified by (3.5) over the interval $-1 \leqslant \eta \leqslant 0$. In this case $\beta_0 = \bar{h}(\eta)$, where $\bar{h}$ is defined in (4.1); and the frequency equations (5.3), after multiplication by $4\xi^{\frac{1}{2}}/3^{\frac{1}{2}}$, and (5.4) reduce respectively to

$$\xi_0 J_1(\xi_0)+\frac{4}{3^{\frac{1}{2}}}\mu[\Omega^2(3-\Omega^2)]^{\frac{1}{2}}[\xi_0 J_0(\xi_0)-\tfrac{1}{2}J_1(\xi_0)] = 0 \tag{5.5a}$$

and

$$J_1(\xi_0) = 0, \tag{5.5b}$$

where the frequency Ω is defined by (3.9b) and where ξ_0 is the value of ξ at the origin, i.e.

$$\xi_0 = \xi|_{\eta=0}. \tag{5.6}$$

For a triangular basin (since the slope is constant), the right-hand side of (5.1) can be integrated and then by (5.6) the value of ξ_0 is given by

$$\xi_0 = 3^{\frac{1}{2}}\mu\{\sin^{-1}(\tfrac{2}{3}\Omega^2-1)+\tfrac{1}{2}\pi\}, \tag{5.7a}$$

and this can be inverted in the form

$$\Omega^2 = \frac{3}{2}\left[1-\cos\frac{\xi_0}{3^{\frac{1}{2}}\mu}\right]. \tag{5.7b}$$

For the antisymmetric mode, from substitution of (5.7a) into (5.5a) an equation is obtained in Ω^2 and the frequency Ω_{Aa} is determined from the lowest root of the latter equation. For the symmetric mode, the lowest root of (5.5*b*) is $\xi_0 = 3.83171$ and when this is substituted in (5.7*b*) we calculate Ω_{As} from

$$\Omega^2_{\mathrm{As}} = \frac{3}{2}\left[1-\cos\frac{3.83171}{3^{\frac{1}{2}}\mu}\right]. \tag{5.8}$$

We now turn to a symmetric parabolic basin specified by (3.20) over the interval $-1 \leqslant \eta \leqslant 0$. In this case $\bar{h}(\eta) = \beta_0(1-\eta)$ by (4.1) and (5.1) and its derivatives become

$$\xi = 3^{\frac{1}{2}}\mu \int_{-1}^{\eta} \left\{(1-\gamma^2)\left[\frac{3}{\Omega^2}-(1-\gamma^2)\right]\right\}^{-\frac{1}{2}} \mathrm{d}\gamma, \tag{5.9}$$

$$\xi_\eta = 3^{\frac{1}{2}}\mu \left\{(1-\eta^2)\left[\frac{3}{\Omega^2}-(1-\eta^2)\right]\right\}^{-\frac{1}{2}}. \tag{5.10}$$

It can be shown by a straightforward calculation that for a parabolic bottom (5.3) takes the form

$$(\xi_0)_\eta\,[\xi_0 J_0(\xi_0) - \tfrac{1}{2}J_1(\xi_0)] = 0$$

and since $(\xi_0)_\eta \neq 0$ we arrive at the following frequency equation for the antisymmetric mode:

$$\xi_0 J_0(\xi_0) - \tfrac{1}{2}J_1(\xi_0) = 0. \tag{5.11}$$

For the symmetric mode the frequency equation is still given by (5.5*b*).

The expression (5.9) is singular at $\eta = 0$; and, in order to extract the lowest root of (5.11) and (5.5*b*) for a parabolic bottom, we need to rewrite the value of ξ at $\eta = 0$ in terms of a suitable representation. This can be effected by writing ξ_0 in terms of the complete elliptic integral of the first kind K (Abramowitz & Stegun 1965, p. 590), i.e.

$$\xi_0 = \mu\Omega K(\tfrac{1}{3}\Omega^2). \tag{5.12}$$

Thus, with the help of (5.12), the values of the frequencies Ω_{Aa} and Ω_{As} are determined, respectively, from the lowest roots of (5.11) and (5.5*b*). The calculated results for both the triangular and parabolic basins were summarized in §3 and are included in tables 1, 2 and 4, 5.

5.2. *A canal of variable section*

Consider a canal of an indefinite extent along its length, with its cross-sectional plane coincident with the (x,z)-plane of the coordinate axes (x,y,z) and its length along the y-direction. For a canal of this kind, the propagation of a travelling wave in the y-direction is discussed in Lamb (1932, p. 366) when the depth of the canal is uniform. We indicate here a generalization of Lamb's result to a canal of variable depth, by a slight reinterpretation of the solution in §4. Thus, in the context of three-dimensional wave motion, consider a wave travelling in the y-direction with velocities u and v given by

$$u(x,y,t) = M\hat{U}(x)\cos(ky-\omega t), \quad v(x,y,t) = M\hat{V}(x)\sin(ky-\omega t). \tag{5.13}$$

After integrating (2.10) with respect to t, apart from the arbitrary function of integration, the amplitude is found to be of the form

$$\phi(x,y,t) = \frac{M}{\omega}\{(h\hat{U}(x))_x + k\hat{V}(x)\}\sin(ky-\omega t). \tag{5.14}$$

An important feature of (5.14) is the dependence of the amplitude on the variation with depth; and, of course, the solution remains valid at an end point of the cross-sectional plane, where the depth is zero.

5.3. *The nature of solutions for other basins*

The application of the general solution in §4 to a basin for which the depth h is zero at one end (corresponding to the end point of a sloping beach) is by now obvious.† In particular, for basins of any profile – not necessarily parabolic – which are symmetric about the z-axis, mathematically it will suffice to consider one-half of the basin profiles (with one end at which $h = 0$) as illustrated in §5.1. However, the solution in §4 may be applied also to basins with two sloping beaches for which the end points at which $h = 0$ are not necessarily symmetric about the z-axis. In order to consider oscillations over a basin with two sloping beaches, it is mathematically more convenient to divide the basin into two separate parts each with a different sloping beach, and then obtain solutions corresponding to each part with the previous procedure. After obtaining these solutions, the final results are obtained with the use of continuity condition at the separation section.

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† The discussion here refers to a two-dimensional situation confined to the (x, z)-plane, but the argument may be generalized to the three-dimensional situation in an obvious manner.

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